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DISCUSSION OF EFFECT OF SKEW ANGLE ON RIGID-FRAME REACTIONS

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DISCUSSION

D. A. NETTLETON,[8] ASSOC. M. ASCE.—An investigation such as this is interesting because it throws light upon a problem of stress analysis that has never been correctly solved by analytical methods.

Table 2 presents a comparison between model results and analytical results for reactions R_x and R_z. The model results for loads at points 1L, 2L, and 3L indicate an increase in R_x for a skew angle of 50° of from 8% to 9% over the values for zero skew. No increase is shown for a load at 4L, and a 1% increase is shown for a load at the center. The analytical results, based on Mr. Rathbun's theory, show R_x to remain constant regardless of the angle of skew. It might be thought that these results confirm the Rathbun theory, within the limits of accuracy of the test. However, the fixed-end moment for a slab of 100-ft span is only 7.5% more than the knee moment for the right-angle rigid frame under investigation. Many engineers believe that increasing the skew angle increases the stiffness of the leg relative to the deck, and therefore that the corner moment approaches the fixed-end moment with increasing skew. Since the maximum possible increase in the corner moment (and therefore R_x) is only 7.5%, it is evident that the variation in results of the test are such that no conclusion can be drawn as to the effect of skew on R_x, or as to the correctness of the analytical theory.

Table 3 presents a comparison between model results and analytical results for the reaction M_x. For a load at the center and for the various skew angles, the model results give values of the order of twenty times greater than the analytical results. It may be thought that the value of M_x is of little use in the design of a rigid-frame bridge, and therefore that this enormous variation is of no importance. However, such is not the case. Consider the eccentricity in vertical reaction represented by the model value of $-$ 4.64 for M_x due to a unit load at the center of the frame skewed at an angle of 50°. The vertical reaction is 0.5, and the eccentricity, therefore, is 9.28 ft toward the obtuse corner. Since the total width of the abutment for this skew is 31.12 ft, the vertical pressure on the foundation at the obtuse corner is more than three times the average pressure. Furthermore, the eccentricity of the vertical reaction increases for sections higher up the leg, due to the effect of R_z. At the top of the leg, the eccentricity is equal to $(-4.64 - 0.743 \times 22) \div 0.5 = 42$ ft. This eccentricity produces a maximum compressive force at the obtuse corner, from axial load, equal to nine times the average compression.

Since the axial force is considered along with bending moment in determining the stresses, this large variation in axial force is of considerable importance. Also, if the axial force varies so much along the width of the section, there is very good reason to believe that the bending moment also varies along the width of the section, probably increasing toward the obtuse corner. Neither the analytical Rathbun method nor the model tests presented by Mr. Boyer will yield

NOTE.—This paper by Walter C. Boyer was published in September, 1950, as *Proceedings-Separate No. 32.* The numbering of footnotes, equations, and tables in this Separate is a continuation of the consecutive numbering used in the original paper.

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any information as to the variation in moment across an internal section. Analytically, this variation could be determined only by some such method as that of "difference equations" and, experimentally, by strain gages along the section.

An experimental determination of internal stresses in a model would be very valuable and should lead to a more correct method of stress analysis than the ones now in use.

MAURICE BARRON,[9] M. ASCE.—The ingenious apparatus and the method of deflection measurement for space structures as developed by the author is a worthy feature of this paper. Space structures have many characteristics that are not manifest in plane structures. The writer has demonstrated[10] that the elastic system about each of the three axes in space may be treated independently when determining the reactions and total stresses. Furthermore, the writer's demonstration shows that the rectangular elastic system is the predominating system and that the other two elastic systems (and the stresses usually associated with them) are of secondary order. Any quantity not determined by the rectangular analysis may be easily found by multiplying a primary value by a simple geometric or elastic constant. When this procedure is not followed and when the three elastic systems are analyzed simultaneously, the structural significance of most elements is obscured.

An examination of the author's original thesis (referred to under the heading "Acknowledgment") shows that the analytical values were determined by numerical solution of four simultaneous elastic equations for each position of load. For each skew angle (0°, 10°, 20°, 30°, 40°, 50°, and 60°) the analysis was repeated, and considerable time was spent on this numerical work. The time required for determining the theoretical values for off-center loading by this method was prohibitive, and, therefore, no comparison of analytical and test values was made.

It must be emphasized that the model chosen is an idealized rigid frame with vertical legs and horizontal deck and that the moment of inertia (I) is uniform for these component parts of the frame. It has been shown[11] that, for this idealized frame, almost all the complicated skew analysis disappears if the method of elastic centers and torsional elastic weights is used. Thus, for vertical loads on the idealized frame, it can be shown that

$$R_x = H \quad \text{(2a)}$$

in which H is the rectangular horizontal reaction on a rectangular frame with equal square span and with the same values of I, and

$$R_z = \epsilon\, R_x = \epsilon\, H \quad \text{(2b)}$$

For vertical loads symmetrically placed, transversely,

$$M_x = \epsilon\,(F - H\,h) \quad \text{(3a)}$$

[9] Cons. Engr., Farkas and Barron, New York, N. Y.

[10] "Reinforced Concrete Skewed Rigid-Frame and Arch Bridges," by Maurice Barron, *Proceedings-Separate No. 13*, ASCE, Vol. 76, April, 1950.

[11] "The Rigid-Frame Bridge," by Arthur G. Hayden and Maurice Barron, John Wiley & Sons, Inc., 3d Ed., New York, N. Y., 1950, p. 149.

in which F is the average of the two fixed-end moments, considering an equivalent rectangular deck as a slab with fixed ends, and in which h is the deck height of the frame. For vertical loads not symmetrically placed, transversely,

$$M_z = \epsilon (F - H h) + P e \qquad (3b)$$

in which P is the vertical load and e is the transverse eccentricity. For all vertical loads

$$M_y = 0 \qquad (4)$$

Applied to skewed rigid frames with rising vaults and to skewed arches, Eqs. 2, 3, and 4 will be correct for all practical design purposes, but for academic accuracy (or for unusual structures) an adjustment of secondary nature will be required.

An examination of the analytical values shown in Tables 2, 3, and 4 will confirm the foregoing statements. Using Eq. 3b, analytical results for off-center loading in Table 3 can easily be determined in order to complete the comparison with model results. When the author determined the analytical results, he used the method[4] of partial summation and the numerical solution of simultaneous elastic equations. At that time no better published method was available. For his idealized rigid frame the elastic weight method yields

$$H = R_x = \frac{\int \frac{M\,m\,ds}{E\,I}}{\int \frac{m\,m\,ds}{E\,I}} \qquad (5)$$

The elastic weight is a uniform load, y/I ($= 22 \times 6/10 = 13.2$ units per linear foot). The numerator of Eq. 5 is the simple-span moment [$= 6.6\,(100\,x - x^2)$] due to this uniform weight. The denominator may be evaluated as follows (E is constant and cancels out):

$$\int \frac{m\,m\,ds}{I} = \int \frac{y^2\,ds}{I} = 2\int_0^{22} \frac{y^2\,dy}{40/3} + \int_0^{100} \frac{22^2\,dl}{10/6} = 29{,}572 \text{ ft}^4$$

or if the steel angles (see Figs. 4 and 6) at the knees are considered to be infinitely stiff (author's assumption):

$$\int \frac{y^2\,ds}{I} = 2\int_0^{20} \frac{y^2\,dy}{40/3} + \int_0^{90} \frac{22^2\,dl}{10/6} = 26{,}536 \text{ ft}^4$$

The author's value for $\Sigma\, y^2/I$, using the mechanical integration (via blocks), is 26,480 ft^4. Therefore, substituting in Eq. 5

$$H = R_x = \frac{6.6\,(100\,x - x^2)}{26{,}480} \qquad (6)$$

Values for R_x ($= H$) from Eq. 6 are presented in Table 5(a), and these values are the same for all skews. Values of R_z ($= \epsilon\, R_x$) are shown for comparison with the values shown in Table 2.

[4] "The Rigid Frame Bridge," by Arthur G. Hayden, John Wiley & Sons, Inc., New York, N. Y., 1941, pp. 137–182.

TABLE 5.—ANALYTICAL VALUES OF CROSS SHEAR R_z AND MOMENT M_x FOR UNIT VERTICAL LOAD.

Skew angle Θ	$\epsilon = \tan\Theta$	(a) Cross Shear R_z;[a] Load at point:					(b) Values of M_x for Eq. 7; Load at point:				
		1	2	3	4	C[b]	1	2	3	4	C[b]
0°	0	0	0	0	0	0					
10°	0.176	0.040	0.070	0.092	0.106	0.110	0.079	0.141	0.185	0.211	0.220
20°	0.364	0.082	0.145	0.191	0.218	0.227	0.164	0.291	0.382	0.436	0.455
30°	0.577	0.130	0.231	0.303	0.346	0.360	0.260	0.461	0.605	0.692	0.721
40°	0.839	0.189	0.335	0.440	0.504	0.524	0.378	0.672	0.965	1.010	1.050
50°	1.192	0.268	0.478	0.626	0.716	0.746	0.537	0.954	1.250	1.430	1.490
R_z (all skews)[c]		0.225	0.400	0.525	0.600	0.625	0.225	0.400	0.525	0.600	0.625

[a] For comparison with the values in Table 2. [b] Center line. [c] By elastic weights.

Using the writer's methods, the following equation may be derived for vertical loads on the idealized rigid frame:

$$M_x = \epsilon \left[22\, R_z - \frac{x\,(L - x)}{2\,L} \right] \quad \text{(7a)}$$

or

$$M_x = \epsilon\, x\,(L - x) \left(\frac{145.2}{26{,}480} - \frac{1}{200} \right) \quad \text{(7b)}$$

Values for Eqs. 7 are shown in Table 5(*b*).

For all vertical loads

$$M_y = 0 \quad \text{(8)}$$

The deviations from zero as shown in Table 4 are the results of small differences that are an inherent flaw in the method of using a numerical solution of simultaneous elastic equations as applied to skew structures.

The writer has repeatedly stressed a common source of error in model analysis. When model results are compared with analytical results and little agreement is shown, it is not proper to condemn either. Further tests and data are required to show which is incorrect. Even when exact agreement is shown, both series of data may be erroneous to the same degree. This warning is particularly suited to space structures. It can be shown that a torsional theory for skewed structures may be incorrect by several hundred percent, and yet an exact correlation will be shown for the rectangular elastic system.

Since the test data for M_x do not agree very well with the analytical results, everything should be suspect rather than merely the formula for F, the torsion factor. Slippage at the hinges or at the knee joints is just as likely. Furthermore, the assumption that the steel angles at the knee are infinitely stiff will bear close scrutiny. This assumption by the author is of little importance for the rectangular system, but the elastic block at the knee is very important for the torsional elastic system. This is easily verified by an examination of the effect the knee section has on the author's analytical results.

Future investigators should find the test apparatus and the method of measurement a great help. Models which would more nearly resemble the rigid-frame bridge could be built easily. Thus, the vault of the model could be cut from large diameter plastic tubes.

The bridges of the future will doubtless use curved-in-plan shapes. The variable skew frame and arch bridge is now coming into prominence as a curved-in-plan structure. Models for these unique, economical, and aesthetic structures could be built easily using a section of large diameter tubing for the curved-in-plan abutment walls. The author's apparatus and method of measurement would be the most logical procedure to adopt.

WALTER C. BOYER,[12] JUN. ASCE.—It is stimulating to observe the revived interest in the skew rigid-frame problem. Within the past few months, three papers about this subject have been published as *Proceedings-Separates* by ASCE.[10,13,14] Each paper has indicated or demonstrated simplified design procedures. The inordinate amount of work entailed in the analysis of skewed structures has been a positive deterrent to the consulting engineer. It is hoped that such recent developments as the simplified procedure of Maurice Barron will serve to countermand such objections.

Mr. Barron's discussion indicates pointedly the objections to the old analysis procedure involving simultaneous equations. Since the equations are quite unbalanced, residue values are evident in some cases (see Table 4), and other values which are normally small may show abnormalities. Values presented by Mr. Barron show closer agreement with model results than those presented by the writer, and since the elastic center permits the isolation of individual effects, these values are considered to be more valid as a comparison.

Mr. Barron has voiced a warning in interpreting the "model analysis" values. He feels that the knee joints or hinges may have been subject to slippage. This phase of the model study was carefully considered and no slippage of consequence could be measured. However, Mr. Barron has emphasized the apparent independency of the rectangular and torsional system. It is this condition that leads the writer to the conclusion that there is no real basis for confidence in the "torsional factor" as now conceived. This problem cannot be resolved, however, without a fundamental study and review of the torsional effect of plates with large width-to-thickness ratios.

Mr. Nettleton has contributed an interesting discussion, and although the writer agrees with his contentions in some respects, it is felt that the model investigation verifies the Rathbun theory in all essentials. The skew rigid frame, with a 100-ft span, was selected intentionally to isolate the effects peculiar to the skew problem. However, the qualitative trends developed indicate the general acceptance of the Rathbun theory for other leg-to-span ratios.

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[13] "Practical Design of Solid-Barrel, Reinforced-Concrete Skew Structures," by Bernard L. Weiner, *Proceedings-Separate No. 39*, ASCE, Vol. 76, October, 1950.

[14] "Laterally Loaded Plane Structures and Structures Curved in Space," by Frank Baron and James P. Michalos, *Proceedings-Separate No. 51*, ASCE, Vol. 77, January, 1951.

It is to be emphasized that the influence of moment (M_x) is almost negligible on other reactive elements. However, this does not imply that its influence on stress distribution across a section is negligible. Mr. Nettleton has indicated that this effect can be quite pronounced in the skew frame. The writer wishes to point out, however, that the same condition is evident in the right frame for loads off the center line of the structure. Mr. Nettleton feels that study should be continued to indicate the variation in stress along the width of section, and the writer is in complete agreement. This is another phase of the skew problem and was not encompassed in the study reported in this paper.